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SOME SIMPLE PROOFS IN
SEMI-MARKOFF PROCESSES

By J. Bram

CNA Research Contribution No. 110

This Research Contribution does not necessarily represent the opinion of the Department of the Navy

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1. Enclosure (1) is forwarded as a matter of possible interest.

2. This Research Contribution shows that by invoking the ergodic theorem, most of the important averages of probabilistic systems that can be represented as semi-Markoff systems can be obtained as limits of ratios, precisely as in the cases where the law of large numbers is applicable. The procedure is illustrated with several examples.

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Erwin Baumgarten
ERWIN BAUMGARTEN

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Operations Evaluation Group

CENTER FOR NAVAL ANALYSES

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By J. Bram

Joseph Bram

February 1969

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ABSTRACT

By invoking the ergodic theorem, most of the important averages of probabilistic systems that can be represented as semi-Markoff systems can be obtained as limits of ratios, precisely as in the cases where the law of large numbers is applicable. The procedure is illustrated with several examples.

This paper is published as a Research Contribution to provide as early as possible the results of CNA supporting analysis to the Naval, Marine, and analytical community.

SEMI-MARKOFF PROCESSES

1. There has recently been some interest in the application of the theory of semi-Markoff processes to problems in ASW. The purpose of this memorandum is to present some simple proofs of the main results.

2. Suppose we are studying a system with a finite number of states 1, 2, ..., K, and that the system changes its state according to a given Markoff structure, with constant transition matrix $A = \{a_{ij}\}$; if x_n and x_{n+1} denote the states at the n^{th} and $(n+1)^{th}$ stages, we have

$$\Pr[x_{n+1} = j | x_n = i] = a_{ij}.$$

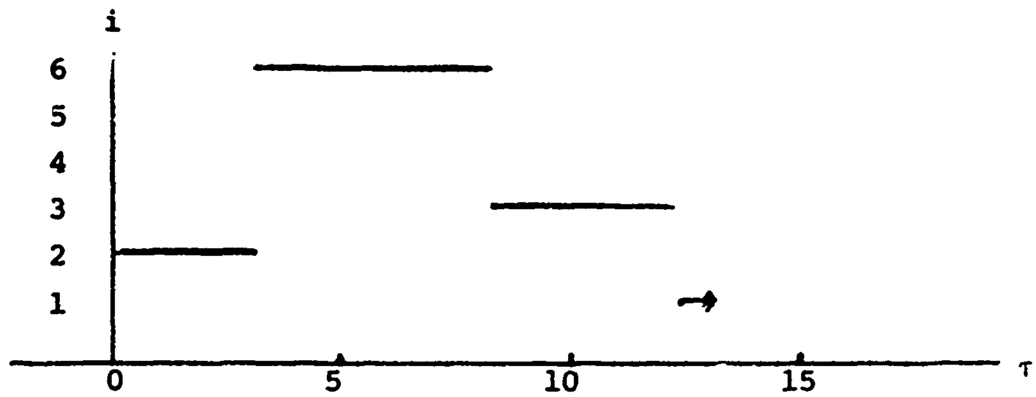
We suppose that there is a unique vector $\langle\pi_i, \ldots, \pi_K\rangle$ with strictly positive components, and $\sum_i \pi_i = 1$, for which

$(\pi_1, \ldots, \pi_K)\, A = (\pi_1, \ldots, \pi_K)$. The π_i are the limiting, or steady-state, probabilities. We also suppose that $a_{ii} = 0$ for $i = 1, \ldots, K$. This structure is fairly common in ASW models, wherein the states refer to search, being in contact, localizing, etc.

3. We now suppose also that we are given waiting-time densities $\varphi_{ij}(w)$, such that if a transition from i to j will take place, the random variable w is chosen from φ_{ij}, and the system stays in state i for w units of time. For simplicity, we assume that w can take on the values 1,2,3,... only (e.g., minutes), but there is no difficulty in extending the results to continuous time. (In some applications, the φ_{ij} depend only on i, not on j.) If we now regard the time-history x_t of states, t=0,1,2,..., we have a random process, but it is not Markoff; the conditional probability of x_{t+1} given the past is not the same as that given only x_t. We

need to know how long the system has been in its present state. Such a structure as this is called a semi-Markoff process.

4. The way to proceed is clear. We take as the state-space, the set of all (i,τ), $i=1,2,\ldots,K$; $\tau=1,2,\ldots$, in which the first coordinate i is one of our original states, and τ denotes the time elapsed from the instant the system entered i to the present time. For example, suppose the first few i's are $i=2,6,3,1$, with waiting times τ = 3, 5, 4. The graph of i versus t is as follows:



The states (i,τ) are: (2,1), (2,2), (2,3), (6,1), (6,2), (6,3), (6,4), (6,5), (3,1), (3,2), (3,3), (3,4), ...

THE ERGODIC THEOREM

5. It is easy to see that the random process (i_t,τ_t) is a Markoff process, for all the information contained in the past (i_1,τ_1), ... (i_n,τ_n) is already contained in (i_n,τ_n), as far as the conditional probability of (i_{n+1},τ_{n+1}) is concerned. Moreover, because of our assumption on the matrix $A = \{a_{ij}\}$, it also follows that our present structure is irreducible, and every state (i,τ) is recurrent. Therefore, if the (unique) limiting probabilities

$$\sigma_{i,\tau}$$

are used as the initial probabilities on (i,τ) for $t = 0$, our process (i_t,τ_t) will be a stationary process, and the "ergodic hypothesis" will be valid; ensemble averages and time averages are equal.

6. More precisely, let Ω denote the space of all sample functions (i_t,τ_t), $t = 0, \pm 1, \pm 2, \ldots$ (extending to the past as well as the future). Let $F(\omega)$ denote a function defined for each sample function ω. Then

$$\int_\Omega F(\omega')\,dP(\omega') = \lim_{n\to\infty} \frac{F(\omega) + F(U\omega) + \ldots + F(U^n\omega)}{n+1} \tag{1}$$

for almost every ω. Here P denotes the probability in Ω, and $U\omega$ is the sample function obtained by shifting ω one time step to the left. As an example, suppose $f(i,\tau)$ is a given function of i and τ. Define $F(\omega)$ on Ω as follows: If

$$\omega = \langle \ldots \quad (i_{-1},\tau_{-1}),\ (i_0,\tau_0),\ (i_1,\tau_1),\ (i_2,\tau_2),\ \ldots \rangle,$$

then put

$$F(\omega) = f(i_0,\tau_0).$$

Since $\int F(\omega)\,dP(\omega) = E[f(i,\tau)]$, we have from (1),

$$E[f] = \lim_{n\to\infty} \left[f(i_0,\tau_0) + f(i_1,\tau_1) + \ldots + f(i_n,\tau_n)\right] \Big/ n+1 \tag{2}$$

7. For the applications, we are usually interested in the limiting value as $t \to \infty$ of some probability or expected value associated with our process (i_t,τ_t). This limiting value, or steady-state value (which is independent of the starting position of (i,τ)) is the same as what we would obtain for any value of t if we started our process with random values of (i,τ) whose probabilities were equal to the limit probabilities of (i,τ) as $t \to \infty$. This implies that our answer is obtainable in Ω, in the form $\int_\Omega F(\omega)\,dP(\omega)$, and then (1) can be invoked.

EXAMPLES

8. We begin with the limiting values

$$\sigma_{i,\tau} = \lim_{t=\infty} \Pr\left[(i_t, \tau_t) = (i, \tau)\right] \tag{3}$$

For this purpose, we define $F(\omega)$ in Ω:

for $\omega = \left\{< i_\nu \ \tau_\nu > : \nu = 0, \pm 1, \nu 2, \ldots\right\}$,

$$F(\omega) = \begin{cases} 1 & \text{if } (i_0, \tau_0) = (i, \tau) \\ 0 & \text{if } (i_0, \tau_0) \neq (i, \tau) \end{cases} \tag{4}$$

Then

$$\sigma_{i,\tau} = \int_\Omega F(\omega)\, dP(\omega)$$

$$= \lim_{n\to\infty} \frac{\xi_0 + \xi_1 + \xi_2 \ldots + \xi_n}{n+1} \tag{5}$$

in which

$$\xi_\nu = \begin{cases} 1 & \text{if } (i_\nu, \tau_\nu) = (i, \tau) \\ 0 & \text{otherwise.} \end{cases}$$

Let us start with $(i_0, \tau_0) = (1,1)$, and let our sample function be of the form

$$(1,1), (1,2), \ldots, (1, w_1), (i_2, 1), (i_2, 2), \ldots (i_2, w_2),$$

$$\ldots (i_k, 1), (i_k, 2), \ldots (i_k, w_k), \ldots$$

with

$n = w_1 + w_2 + \ldots + w_k$. There are k possibly different i_n's in this segment $0 < t \leq n$, and each is repeated w_m times. We will have roughly $k\pi_i$ of them equal to i (cf. paragraph 2); for each i_m that equals i, we will have exactly one $\tau_\nu = \tau$ if and only if $w_m \geq \tau$, and none if $w_m < \tau$. Now, given that $i_m = i$,

the waiting time density in state i is

$$\varphi_i(w) = \sum_j \varphi_{ij}(w)\, a_{ij} \quad . \tag{6}$$

So the probability that $\tau \le w_m$, given $i_m = i$,

is

$$\sum_{w'=\tau}^{\infty} \varphi_i(w') \overset{\text{def}}{=} E_i(\tau) \tag{7}$$

It follows that roughly $k\pi_i E_i(\tau)$ of the ξ's in (5) equal 1, and the quotient in (5) is about

$$\frac{k\pi_i E_i(\tau)}{n} = \frac{\pi_i E_i(\tau)}{(w_1 + w_2 + \ldots + w_k)/k} \tag{8}$$

In the denominator, we have in the sum $w_1 + \ldots + w_k$ about $k\pi_1$ w's from $\varphi_1(w)$, $k\pi_2$ w's from $\varphi_2(w)$, etc. Let

$$\bar{w}_i = \sum_{w=1}^{\infty} \varphi_i(w)\, w \ ,$$

the mean wait-time in state i. Then we have

$$w_1 + \ldots. + w_k \sim k\pi_1 \bar{w}_1 + k\pi_2 \bar{w}_2 + \ldots + k\pi_K \bar{w}_K \ ,$$

and (5) and (8) now give

$$\sigma_{i\tau} = \frac{\pi_i E_i(\tau)}{\pi_1 \bar{w}_1 + \ldots + \pi_K \bar{w}_K} \quad ,$$

i.e.,

$$\sigma_{i,\tau} = \frac{\pi_i E_i(\tau)}{\sum_j \pi_j \bar{w}_j} \tag{9}$$

This is an important result.

9. We may also require

$$\sigma_i = \lim_{t=\infty} \Pr\left[i_t = i\right] \tag{10}$$

Evidently, we have

$$\sigma_i = \sum_\tau \sigma_{i,\tau} \quad .$$

We use (9) and observe that

$$\sum_{\tau=1}^{\infty} E_i(\tau) = \sum_{w=1}^{\infty} w\varphi_i(w) = \bar{w}_i \tag{11}$$

so that

$$\sigma_i = \frac{\pi_i \bar{w}_i}{\sum_j \pi_j \bar{w}_j} \tag{12}$$

This is also important. Note the difference between π_i and σ_i.

10. In a forthcoming memo, D. Culbertson defines his states not as we do, but by (i,y), in which i has the same meaning as ours, but y denotes the time remaining in state i until the next jump. Using our set-up, take y fixed and i fixed, and define $F(\omega)$ in Ω thus: if

$$\omega = < \ldots . < i_0, \tau_0 >, < i_0, \tau_0 + 1 >, \ldots < i_0, w >, < i_1, 1 > \ldots . >$$

$$\text{put} \quad F(\omega) = \begin{cases} 1 & \text{if } w - \tau_0 + 1 = y \text{ and } i_0 = i \\ 0 & \text{otherwise .} \end{cases}$$

Then $F(\omega) = 1$ if and only if the system is in state i and the time remaining at $t = 0$ until a new state is reached equals y. Then

$$\int_\Omega F(\omega)\,dP(\omega) = \Pr\left[i_t = i \;\&\; y_t = y\right] \overset{\text{def}}{=} \tilde{\sigma}_{i,y} \qquad (13)$$

From (1), we have

$$\tilde{\sigma}_{i,y} = \lim_{n\to\infty} \frac{\xi_1 + \xi_2 + \ldots + \xi_n}{n} \qquad (14)$$

in which each

$$\xi_\nu = \begin{cases} 1 & \text{if } i_\nu = i \;\&\; w_m - \tau_\nu + 1 = y \\ 0 & \text{otherwise} \end{cases}$$

in which w_m is the waiting-time for the currently occupied state. Now, given that $i_\nu = i$, let w denote the waiting time for the current sojourn in this state. Then the τ's that we would observe are $\tau = 1,2,\ldots w$ while the y's are $y = w, w-1,\ldots,1$; i.e., exactly one of the y's will equal our specified y if and only if one of the τ's does. Therefore the argument given in paragraph 8. holds here also, and we get

$$\tilde{\sigma}_{i,y} = \frac{\pi_i E_i(y)}{\sum_j \pi_j \bar{w}_j} \quad ; \qquad (15)$$

the (i,τ)'s and (i,y)'s have exactly the same distribution.

11. Another question that is often of interest concerns the mean return time to state 1, given that the system has just left state 1. By examining a few typical sample functions (and their graphs), we quickly see that

$$E\,[\text{return time to state 1} \mid \text{system just left state 1}]$$

$$= \frac{\int_\Omega F\, dP}{\int_\Omega G\, dP} \tag{16}$$

in which

$$F(\omega) = \begin{cases} \text{smallest positive } t \text{ for which } i_t = 1 \text{ if } i_0 = 1 \ \&\ i_1 \neq 1 \\ 0 \quad \text{otherwise} \end{cases}$$

and

$$G(\omega) = \begin{cases} 1 \quad \text{if } i_0 = 1 \ \&\ i_1 \neq 1 \\ 0 \ \text{otherwise} . \end{cases}$$

By (1), we obtain

$$E\,[\text{return time} \mid \text{just left}] = \lim_{n\to\infty} \frac{t_1 + t_2 + \ldots .t_n}{\xi_1 + \xi_2 + \ldots + \xi_n} \tag{17}$$

in which the t_ν and ξ_ν are obtainable from any one sample function. Most of the t_ν and ξ_ν are zero. It is only when we have an $i_\nu = 1$ and $i_{\nu+1} \pm 1$ that $t_\nu > 0$ and $\xi_\nu = 1$. So (17) can be written:

$$E[\text{return time} \mid \text{just left}] = \lim_{k\leftarrow\infty} \frac{W_1 + W_2 + \ldots + W_k}{k} \tag{18}$$

in which k denotes the number of times we observe the system leave state 1, up to time n, and W_m is the mth return time. Let r denote the number of actual transitions (a change in i) that occur up to time n. Then roughly $k = r\pi_1$ = no. of times i = 1; ...$r\pi_K$ = no. of times i = K. Therefore the numerator in (18) is roughly $r\pi_2\overline{w}_2 + r\pi_3\overline{w}_3 + \ldots r\pi_K\overline{w}_K$ and since $k = \pi_1 r$, (18) becomes

$$E\,[\text{return time} \mid \text{just left } i=1]$$

$$= \frac{\pi_2\overline{w}_2 + \ldots + \pi_K\overline{w}_K}{\pi_1\,\overline{w}_1}\quad \overline{w}_1$$

$$= \frac{\sigma_2 + \sigma_3 \ldots + \sigma_K}{\sigma_1}\quad \overline{w}_1 \qquad (19)$$

using (12). For any state i, the mean recurrence time is

$$E[\text{return time} \mid \text{just left } i] = \frac{1 - \sigma_i}{\sigma_i}\;\overline{w}_i\,. \qquad (20)$$

12. Our final example concerns the observed waiting time distribution for a given state, say i = 1. When the steady-state condition prevails, we observe our system, and given that i = 1, we inquire when it arrived in i = 1 and when does it leave, and call the difference, departure time minus arrival time, the wait-time (or sojourn time). The distribution of this variable is not the same as the given wait time distribution. We denote this variable by w, and we calculate its characteristic function $E[e^{j\alpha w}]$.

Put
$$F(\omega) = \begin{cases} e^{j\alpha w} & \text{if } i_0 = 1 \\ 0 & \text{if } i_0 \neq 1 \end{cases}$$

and
$$G(\omega) = \begin{cases} 1 & \text{if } i_0 = 1 \\ 0 & \text{if } i_0 \pm 1 \end{cases}$$

Then
$$E[e^{j\alpha w} \mid i = 1] = \frac{\int_\Omega F \, dP}{\int_\Omega G \, dP}$$

and by use of (1), and arguing as in the last paragraph, we have

$$E\left[e^{j\alpha w} \mid i=1\right] = \lim_{k\to\infty} \frac{w_1 e^{j\alpha w_1} + \ldots + w_k e^{j\alpha w_k}}{w_1 + \ldots + W_k}$$

in which $w_1, \ldots, W_k$ denote k sojourn times for i = 1. Dividing numerator and denominator by k and going to the limit, we get

$$E\left[e^{j\alpha w} \mid i = 1\right] = \frac{E[we^{j\alpha w} \mid i = 1]}{\overline{w}}$$
$$= \int e^{j\alpha w} \frac{w\varphi_1(w)\,dw}{\overline{w}} \quad ; \quad (21) .$$

it follows from (21), that the distribution we are seeking is given by the density function

$$\frac{w\varphi_1(w)}{\overline{w}} \quad . \quad (22)$$

In case $\varphi_1(w)$ has the form $\varphi_1(w) = \lambda e^{-\lambda w}$; the true mean holding time is $\overline{w} = \lambda^{-1}$; but by (22), what we observe if we use the procedure outlined at the beginning is $\overline{w} = 2\lambda^{-1}$. This is the so-called "inspection paradox."

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